Integrals-tasks (VII part)
Integration of some trigonometric functions

We'll give you tips for four types of integrals of trigonometric functions.

A) Integral type $\int R(\sin x, \cos x)dx$

These are integrals in which the *sinx* and *cosx* do not have degrees . Here we introduce substitution $\boxed{tg\frac{x}{2}=t}$

From substitution we get : (using the formula from trigonometry)

$$\sin x=\frac{\sin x}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cancel{\cos^2\frac{x}{2}}\left(2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)}{\cancel{\cos^2\frac{x}{2}}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1\right)}=\frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1}=\frac{2t}{t^2+1}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos x}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cancel{\cos^2\frac{x}{2}}\left(1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)}{\cancel{\cos^2\frac{x}{2}}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1\right)}=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

As $tg\frac{x}{2}=t$ then $\frac{x}{2}=arctgt \to x=2arctgt \to dx=\frac{2}{1+t^2}dt$

To summarize:

When we take substitution $\boxed{tg\frac{x}{2}=t}$ we have:

$$\boxed{\begin{aligned}\sin x&=\frac{2t}{1+t^2}\\ \cos x&=\frac{1-t^2}{1+t^2}\\ dx&=\frac{2}{1+t^2}dt\end{aligned}}$$

Substitution $tg\frac{x}{2}=t$ is *universal trigonometric substitution* and can always be used, but it's easier, depending on the look of function to use the following substitution:

B) Integral type $\int R(tgx)dx$ and $\int R(\sin^2 x, \cos^2 x, \sin x \cdot \cos x)dx$

These are integrals where the function under the integral can be reduced to tgx or that are occurring degree of sinus and cosine and product $\sin x \cdot \cos x$

Here we introduce substitution **:** $\boxed{tgx = t}$

From substitution we get : (using the formula from trigonometry)

$$\sin^2 x = \frac{\sin^2 x}{1} = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{tg^2 x}{tg^2 x + 1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x}{1} = \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\frac{\cos^2 x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{1}{tg^2 x + 1} = \frac{1}{1+t^2}$$

$$\sin x \cdot \cos x = \frac{\sin x \cdot \cos x}{1} = \frac{\sin x \cdot \cos x}{\sin^2 x + \cos^2 x} = \frac{\frac{\sin x \cdot \cancel{\cos x}}{\cos^{\cancel{2}} x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} = \frac{tgx}{tg^2 x + 1} = \frac{t}{1+t^2}$$

$$tg\frac{x}{2} = t \to \frac{x}{2} = arctgt \to x = 2arctgt \to dx = \frac{2}{1+t^2}dt$$

To summarize:

When we take substitution $\boxed{tgx = t}$ we have :

$$\boxed{\begin{aligned} &\sin^2 x = \frac{t^2}{1+t^2} \\ &\cos^2 x = \frac{1}{1+t^2} \\ &\sin x \cdot \cos x = \frac{t}{1+t^2} \\ &dx = \frac{2}{1+t^2}dt \end{aligned}}$$

C) Integral type $\int \sin^m x \cdot \cos^n x dx$

Two situations:

i) If m and n are integers

ii) If m and n are rational numbers

In both situations we introduce substitution $\boxed{\sin x = u}$ or $cosx=u$ but,

i) when m and n are integers integral is reduced to integration of rational function

ii) when m and n are rational numbers we have integral of differential binomial

D) Integral type $\int \sin ax \cos bx dx; \int \sin ax \sin bx dx; \int \cos ax \cos bx dx;$

First, use the trigonometric formulas:

$sinax\ sinbx = \frac{1}{2}[cos(a-b)x - cos(a+b)x]$

$sinax\ cosbx = \frac{1}{2}[sin(a+b)x + sin(a-b)x]$

$cosax\ cosbx = \frac{1}{2}[cos(a+b)x + cos(a-b)x]$

Some "tricks":

If in integral we have expression $\sqrt{a^2 - x^2}$, it is convenient to take ***x = asint*** as it "destroyed " the root:

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin t)^2} = \sqrt{a^2 - a^2(\sin t)^2} = a\sqrt{1 - \sin^2 t} = a \cos t$$

If in integral we have expression $\sqrt{x^2 + a^2}$, it is convenient to take ***x=a tgt*** as it "destroyed " the root:

$$\sqrt{x^2 + a^2} = \sqrt{(atgt)^2 + a^2} = \sqrt{a^2tg^2t + a^2} = a\sqrt{tg^2t + 1} = a\sqrt{\frac{\sin^2 t}{\cos^2 t} + 1} = a\sqrt{\frac{\sin^2 t + \cos^2 t}{\cos^2 t}} = a\sqrt{\frac{1}{\cos^2 t}} = a\frac{1}{\cos t}$$

Examples

Example 1. $\int \frac{dx}{\sin x} = ?$

This integral we have already solved in **Integrals-tasks (I part)** without using trigonometric substitution.

Now, we can use $tg\frac{x}{2} = t$. So : $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$

$$\int \frac{dx}{\sin x} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}}dt = \int \frac{1}{t}dt = \ln|t| + C = \ln\left|tg\frac{x}{2}\right| + C$$

Example 2. $\int \frac{2-\sin x}{2+\cos x}dx = ?$

Here we use $tg\frac{x}{2} = t$ because *sinx* and *cosx* have no degrees…

$$tg\frac{x}{2} = t$$

$$dx = \frac{2}{1+t^2}dt$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos t = \frac{1-t^2}{1+t^2}$$

$$\int \frac{2-\sin x}{2+\cos x}dx = \int \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt = \int \frac{\frac{2+2t^2-2t}{1+t^2}}{\frac{2+2t^2+1-t^2}{\cancel{1+t^2}}} \cdot \frac{2}{\cancel{1+t^2}}dt = 4\int \frac{t^2-t+1}{(t^2+3)(1+t^2)}dt$$

This is integral of rational function …

Beware: both terms in the denominator are irresolvable:

So:

$$\frac{t^2-t+1}{(t^2+3)(1+t^2)}=\frac{At+B}{t^2+3}+\frac{Ct+D}{1+t^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots / \cdot(t^2+3)(1+t^2)$$

$$t^2-t+1=At+At^3+B+Bt^2+Ct^3+3Ct+Dt^2+3D$$

$$t^2-t+1\equiv(A+C)t^3+(B+D)t^2+(A+3C)t+B+3D$$

$$A+C=0,\ B+D=1,\ A+3C=-1 \text{ i } B+3D=1$$

$$A=1/2,\ B=1,\ C=-1/2,\ D=0$$

$$4\int\frac{t^2-t+1}{(t^2+3)(1+t^2)}dt=4\int\frac{\frac{1}{2}t+1}{t^2+3}dt+4\int\frac{\frac{-1}{2}t}{1+t^2}dt=2\int\frac{t+2}{t^2+3}dt-2\int\frac{t}{1+t^2}dt$$

Solution is:

$$=\ln\frac{t^2+3}{1+t^2}+\frac{2}{\sqrt{3}}arctg\frac{t}{\sqrt{3}}+C=\boxed{\ln\frac{3+tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}+\frac{2}{\sqrt{3}}arctg\frac{tg\frac{x}{2}}{\sqrt{3}}+C}$$

Example 3. $I = \int \frac{dx}{(2+\cos x)\sin x} = ?$

Often integral to give a letter mark, most frequently used letters : *I, J* …

Sabstitution: $tg\frac{x}{2} = t$.

$$tg\frac{x}{2} = t$$
$$dx = \frac{2}{1+t^2}dt$$
$$\sin x = \frac{2t}{1+t^2}$$
$$\cos t = \frac{1-t^2}{1+t^2}$$

So: $$I = \int \frac{\frac{2}{1+t^2}}{\left(2+\frac{1-t^2}{1+t^2}\right)\cdot\frac{2t}{1+t^2}}dt = \int \frac{dt}{\frac{2+2t^2+1-t^2}{1+t^2}\cdot t} = \int \frac{1+t^2}{(t^2+3)t}dt$$

Now, we do:

$$\frac{1+t^2}{(t^2+3)t} = \frac{A}{t} + \frac{Bt+C}{t^2+3} = \frac{At^2+3A+Bt^2+Ct^2}{t(t^2+3)}$$
$$1+t^2 = A(t^2+3)+(Bt+C)t$$
$$1+t^2 = At^2+3A+Bt^2+Ct$$
$$1+t^2 = t^2(A+B)+Ct+3A$$

$$A+B=1, \quad C=0, \quad 3A=1$$

$$A=1/3, \quad B=2/3, \quad C=0$$

$$I = \frac{1}{3}\int\frac{dt}{t} + \frac{2}{3}\int\frac{t}{t^2+3}dt = \left| \begin{matrix} t^2+3=u \\ tdt = \frac{1}{2}du \end{matrix} \right| = \frac{1}{3}\ln|t| + \frac{1}{3}\int\frac{du}{u}+c$$

$$I = \frac{1}{3}\ln|t| + \frac{1}{3}\ln(t^2+3)+C = \boxed{\frac{1}{3}\ln\left|tg\frac{x}{2}\right| + \frac{1}{3}\ln\left(tg^2\frac{x}{2}+3\right)+C}$$

Example 4. $I=\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx=?$

$tgx=t$ So:

$$\boxed{\begin{array}{l}\sin^2 x=\frac{t^2}{1+t^2}\\ \cos^2 x=\frac{1}{1+t^2}\\ \sin x\cdot\cos x=\frac{t}{1+t^2}\\ dx=\frac{2}{1+t^2}dt\end{array}}$$

$$I=\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx=\int \frac{\frac{t}{1+t^2}}{\left(\frac{t^2}{1+t^2}\right)^2+\left(\frac{1}{1+t^2}\right)^2}\cdot\frac{2}{1+t^2}dt=\int \frac{\frac{t}{\cancel{1+t^2}}}{\frac{t^4+1}{\cancel{(1+t^2)}^{\cancel{2}}}}\frac{2}{\cancel{1+t^2}}dt=$$

$$=\int \frac{2t}{t^4+1}dt=\int \frac{2t}{(t^2)^2+1}dt\left|\begin{array}{l}t^2=z\\ 2tdt=dz\end{array}\right|=\int \frac{dz}{z^2+1}=arctgz+C=arctgt^2+C=arctg(tg^2x)+C$$

This task we could solve in another way, using trigonometric formulas

The idea is to transform the expression in the denominator. Start from the basic identity:

$\sin^2 x+\cos^2 x=1$............................$/()^2$

$\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x=1$

$\sin^4 x+\cos^4 x=1-2\sin^2 x\cos^2 x$

$\sin^4 x+\cos^4 x=1-\frac{4\sin^2 x\cos^2 x}{2}$

$\sin^4 x+\cos^4 x=1-\frac{\sin^2 2x}{2}$

$\sin^4 x+\cos^4 x=\frac{2-\sin^2 2x}{2}$

$\sin^4 x+\cos^4 x=\frac{1+\boxed{1-\sin^2 2x}}{2}$

$\boxed{\sin^4 x+\cos^4 x=\frac{1+\cos^2 2x}{2}}$

Return to the integral:

$$I=\int\frac{\sin x\cos x}{\frac{1+\cos^2 2x}{2}}dx=\int\frac{2\sin x\cos x}{1+\cos^2 2x}dx=\int\frac{\sin 2x}{1+\cos^2 2x}dx\left|\begin{array}{l}\cos 2x=t\\ -\sin 2x\cdot 2dx=dt\\ \sin 2xdx=\frac{dt}{-2}\end{array}\right|=\frac{1}{-2}\int\frac{dt}{1+t^2}$$

$$=\frac{1}{-2}arctgt+C=\boxed{-\frac{1}{2}arctg(\cos 2x)+C}$$

Example 5. $\int\sqrt{a^2-x^2}=?$

Remember this integral?

We solved it so far in two ways: a partial integration and using Ostrogradski formula…

Here's another way of solving:

If we use sabstitution $x=a\sin t$

$$\sqrt{a^2-x^2}=\sqrt{a^2-(a\sin t)^2}=\sqrt{a^2-a^2(\sin t)^2}=a\sqrt{1-\sin^2 t}=a\cos t$$

$$x=a\sin t\to dx=a\cos tdt$$

$$\int\sqrt{a^2-x^2}=\int a\cos t\cdot a\cos tdt=a^2\int\cos^2 tdt=a^2\int\frac{1+\cos 2t}{2}dt=$$

$$\frac{a^2}{2}\int(1+\cos 2t)dt=\frac{a^2}{2}(t+\frac{1}{2}\sin 2t)+C=$$

From substitution , we have:

$$x=a\sin t\to\sin t=\frac{x}{a}\to t=\arcsin\frac{x}{a}$$ go back...

$$\frac{1}{2}\sin 2t=\frac{1}{\not{2}}\not{2}\sin t\cos t=\sin t\cdot\sqrt{1-\sin^2 t}=\sin(\arcsin\frac{x}{a})\sqrt{1-\sin^2(\arcsin\frac{x}{a})}=$$

$$=\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}=\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}=\frac{x}{a^2}\sqrt{a^2-x^2}$$

Solution is: $\frac{a^2}{2}(t+\frac{1}{2}\sin 2t)+C=\frac{a^2}{2}(\arcsin\frac{x}{a}+\frac{x}{a^2}\sqrt{a^2-x^2})+C=\boxed{\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C}$